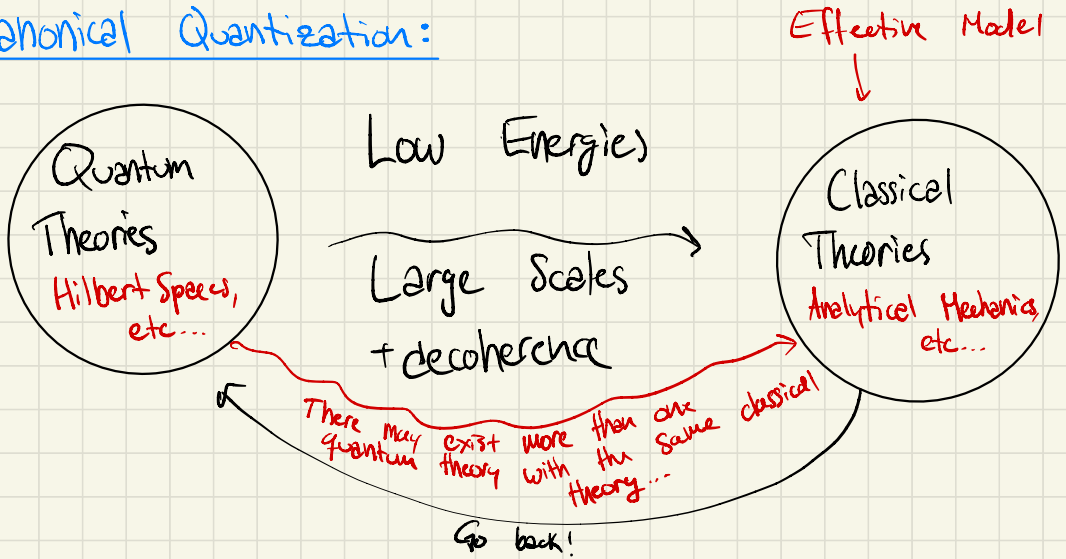


Lecture 4: Field Quantization

Content: Canonical Quantization, Field Quantization, Diagonalization, Klein-Gordon Hamiltonian

Canonical Quantization:



Quantization $\equiv$ "Educated guess"

For now: quadratic H case "solved".

Recall: Standard quantization approach:

Suppose we have N degrees of freedom (classical)

$\hookrightarrow$ position & momentum q_j, p_j ($j=1, \dots, N$)

where $\{q_j, p_k\} = \delta_{jk}$. (4.1)

Hamiltonian:

$$H = \sum_{j=1}^N \frac{p_j^2}{2m} + \frac{1}{2} m \sum_{j,k=1}^N q_j [\bar{Q}]_{j,k} q_k \quad (4.2)$$

Symmetric $N \times N$



Some "Spring-like" potential: H.O.

Now, what is the quantum $\hat{H}$ such that (4.2) is obtained via the classical limit?

Canonical Quantization:

$$[i] \quad (q_j, p_j) \longrightarrow (\hat{q}_j, \hat{p}_j)$$

$$[ii] \quad \delta_{jk} = \{q_j, p_k\} \longrightarrow [\hat{q}_j, \hat{p}_k] = i \delta_{jk}$$

$$[iii] \quad H(q, p) \longrightarrow \hat{H}(\hat{q}, \hat{p})$$

Worse example: $H = qp$? Quantum mechanically, $\hat{q} \hat{p}$ is

not Hermitian: $\hat{H} \rightarrow \frac{1}{2} (\hat{q} \hat{p} + \hat{p} \hat{q})$?

$$\rightarrow \sqrt{\hat{q}} \hat{p} \sqrt{\hat{q}}?$$

Solution: these Hamiltonians are not commonly encountered.

Harmonic Oscillator:

$$\hat{H} = \sum_{j=1}^N \frac{\hat{p}_j^2}{2m} + \frac{1}{2} m \sum_{j,k=1}^N \hat{q}_j [\bar{\bar{Q}}]_{jk} \hat{q}_k \quad (4.3)$$

(Symmetric)

Where # dof $N \rightarrow \infty$ in quantum systems.

Much experience shows this "works".

For fields: "works" $\Rightarrow$ expressed quantized (unitary) representation of Poincaré Group.

How to solve Hamiltonian (QHO) in eq. (4.3)?

Since $[\bar{\bar{Q}}]_{jk}$ is symmetric,

$$\bar{\bar{Q}}^T = \bar{\bar{Q}} \Rightarrow \exists \bar{O} \text{ s.t. } \bar{O}^T \bar{O} = \mathbb{I}$$

$$\text{and } \bar{O} \bar{\bar{Q}} \bar{O}^T = \bar{D}$$

where $\bar{D}$ is some diagonal matrix.

$$\text{Let } \hat{q}'_j = \sum_{k=1}^N [\bar{O}]_{jk} \hat{q}_k \quad (4.4)$$

$$\hat{p}'_j = \sum_{k=1}^N [\bar{O}]_{jk} \hat{p}_k \quad (4.5)$$

where $\hat{q}'_j, \hat{p}'_j$ preserve the canonical commutation relation.

Proof: We know $[\hat{q}_j, \hat{p}_k] = i \delta_{jk}$.

Then:

$$\begin{aligned}
 [\hat{q}_j, \hat{p}_k] &= \left[\sum_{n=1}^N [\bar{O}]_{jn} \hat{q}_n, \sum_{m=1}^N [\bar{O}]_{km} \hat{p}_m \right] \\
 &= \sum_{n=1}^N \sum_{m=1}^N \left([\bar{O}]_{jn} \hat{q}_n [\bar{O}]_{km} \hat{p}_m - [\bar{O}]_{km} \hat{p}_m [\bar{O}]_{jn} \hat{q}_n \right) \\
 &= \sum_{n=1}^N \sum_{m=1}^N [\bar{O}]_{jn} [\bar{O}]_{km} [\hat{q}_n, \hat{p}_m] \\
 &\quad \text{(since } \bar{O} \text{ matrix entries are just numbers)} \\
 &= i \sum_{n=1}^N \sum_{m=1}^N [\bar{O}]_{jn} [\bar{O}]_{km} \delta_{nm} \\
 &= i \sum_{n=1}^N [\bar{O}]_{jn} [\bar{O}]_{nk} \\
 &\quad \text{(since } \bar{O} \text{ symmetric)} \\
 &= i \sum_{n=1}^N (\mathbb{I}_{nn}) \delta_{jk} \\
 &\quad \text{(since } \bar{O}^T \bar{O} = \bar{O} \bar{O}^T = \mathbb{I}) \\
 &= i \delta_{jk}
 \end{aligned}$$

As required. 

Substituting these new variables eq. (4.4), (4.5) into

$\hat{H}$ (4.3), we have that

$$\hat{H} = \sum_{j,k=1}^N \frac{\hat{p}_{jk}^2}{2m} + \frac{m}{2} \sum_{j,k=1}^N \sum_{l,m=1}^N \hat{q}_l' \overbrace{[\bar{O}]_{jl} [\bar{Q}]_{jk} [\bar{O}^T]_{km}}^{\bar{O}} \hat{q}_m'$$

$$= \sum_{j=1}^N \frac{\hat{p}_j^2}{2m} + \frac{m}{2} \sum_{\lambda=1}^N \omega_{\lambda}^2 \hat{q}_{\lambda}^2 \quad (4.6)$$

where the ω_{λ}^2 are the **Eigenvalues** of $\bar{\bar{D}}$:

$$\bar{\bar{D}} = \begin{pmatrix} \omega_1^2 & & 0 \\ & \omega_2^2 & \\ 0 & & \ddots \end{pmatrix}. \quad (4.7)$$

$\Rightarrow$ **Diagonalize** via **Ladder Operators**:

$$\hat{\hat{a}}_{\lambda} = \sqrt{\frac{m\omega_{\lambda}}{2}} \left(\hat{q}_{\lambda}' + \frac{i}{m\omega_{\lambda}} \hat{p}_{\lambda}' \right) \quad (4.8)$$

$$\hat{\hat{a}}_{\lambda}^{\dagger} = \sqrt{\frac{m\omega_{\lambda}}{2}} \left(\hat{q}_{\lambda}' - \frac{i}{m\omega_{\lambda}} \hat{p}_{\lambda}' \right) \quad (4.9)$$

$$\text{THEN } \hat{H} = \sum_{\lambda=1}^N \omega_{\lambda} \left(\hat{\hat{a}}_{\lambda}^{\dagger} \hat{\hat{a}}_{\lambda} + \frac{1}{2} \right) \quad (4.10)$$

Proof: First, note that

$$\hat{q}_{\lambda}' = \frac{1}{\sqrt{2m\omega_{\lambda}}} (\hat{\hat{a}}_{\lambda} + \hat{\hat{a}}_{\lambda}^{\dagger}) \quad \text{and}$$

$$\hat{p}_{\lambda}' = \frac{1}{i} \sqrt{\frac{m\omega_{\lambda}}{2}} (\hat{\hat{a}}_{\lambda} - \hat{\hat{a}}_{\lambda}^{\dagger}). \quad (\text{This is easy to show})$$

The corresponding commutation relations are given by

$$\begin{aligned}
[\hat{a}_e, \hat{a}_m^\dagger] &= \frac{m\omega_e}{2} \left[\hat{q}_e' + \frac{i}{m\omega_e} \hat{p}_e', \hat{q}_m' - \frac{i}{m\omega_e} \hat{p}_m' \right] \\
&= \frac{m\omega_e}{2} \left([\hat{q}_e', \hat{q}_m'] - \frac{i}{m\omega_e} [\hat{q}_e', \hat{p}_m'] \right. \\
&\quad \left. + \frac{i}{m\omega_e} [\hat{p}_e', \hat{q}_m'] + \frac{1}{m^2\omega_e^2} [\hat{p}_e', \hat{p}_m'] \right) \\
&= \frac{m\omega_e}{2} \left(-\frac{2i}{m\omega_e} \cdot i \delta_{em} \right) \\
&= \delta_{em}
\end{aligned}$$

OR

$$[\hat{a}_e, \hat{a}_e^\dagger] = 1$$

with $\hat{a}_e^\dagger \hat{a}_e = \hat{N}_e$ (Number operator)

Furthermore, we find that

$$\begin{aligned}
\hat{q}_e'^2 &= \frac{1}{2m\omega_e} (\hat{a}_e + \hat{a}_e^\dagger) (\hat{a}_e + \hat{a}_e^\dagger) \\
&= \frac{1}{2m\omega_e} (\hat{a}_e^2 + \hat{a}_e \hat{a}_e^\dagger + \hat{a}_e^\dagger \hat{a}_e + \hat{a}_e^{\dagger 2}) \\
&= \frac{1}{2m\omega_e} (\hat{a}_e^2 + \hat{a}_e^{\dagger 2} + 1 + 2\hat{a}_e^\dagger \hat{a}_e)
\end{aligned}$$

$$\begin{aligned}
\hat{p}_e'^2 &= -\frac{m\omega_e}{2} (\hat{a}_e - \hat{a}_e^\dagger) (\hat{a}_e - \hat{a}_e^\dagger) \\
&= -\frac{m\omega_e}{2} (\hat{a}_e^2 + \hat{a}_e^{\dagger 2} - \hat{a}_e \hat{a}_e^\dagger - \hat{a}_e^\dagger \hat{a}_e) \\
&= -\frac{m\omega_e}{2} (\hat{a}_e^2 + \hat{a}_e^{\dagger 2} - 1 - 2\hat{a}_e^\dagger \hat{a}_e)
\end{aligned}$$

Then, since

$$\begin{aligned}
\hat{H} &= \sum_{j=1}^N \frac{\hat{p}_j^2}{2m} + \frac{M}{2} \omega_j^2 \hat{q}_j^2 \\
&= \sum_{j=1}^N -\frac{\omega_j}{4} (\hat{a}_j^2 + \hat{a}_j^{+2} - 1 - 2\hat{a}_j^+ \hat{a}_j) \\
&\quad + \frac{\omega_j}{4} (\hat{a}_j^2 + \hat{a}_j^{+2} + 1 + 2\hat{a}_j^+ \hat{a}_j) \\
&= \sum_{j=1}^N \frac{\omega_j}{4} (2 + 4\hat{a}_j^+ \hat{a}_j) \\
&= \sum_{j=1}^N \omega_j \left(\frac{1}{2} + \hat{a}_j^+ \hat{a}_j \right).
\end{aligned}$$



Canonical Quantization (Fields):

$$(\phi(x), \pi(x)) \longrightarrow (\hat{\phi}(x), \hat{\pi}(x))$$

Classically, for equal time fields,

$$\{\phi(x), \pi(y)\} = \delta^{(3)}(x-y)$$

$$\longrightarrow [\hat{\phi}, \hat{\pi}] = i\delta^{(3)}(x-y)$$

As in last lecture, the Hamiltonian for the Klein-Gordon field is

$$H_{KG} = \frac{1}{2} \int d^3x \left\{ \pi^2(x) + (\nabla \phi(x))^2 + m^2 \phi^2(x) \right\} \quad (4.11)$$

which goes over into

$$\hat{H}_{KG} = \frac{1}{2} \int d^3x \left\{ \hat{\pi}^2(x) + (\nabla \hat{\phi}(x))^2 + m^2 \hat{\phi}^2(x) \right\} \quad (4.12)$$

Furthermore,

↑
similar to discrete 'Q'
matrix in discrete QHO!

$$\begin{array}{ccc} \text{discrete} & & \text{Continuous} \\ \text{label} & j & \text{label} \quad x \\ & \longleftrightarrow & \\ q_j & & q_x \equiv \phi(x) \end{array}$$

However, Hamiltonian quantization only gives one subset of the Poincaré group, and not all of it.

Diagonalization:

From discrete variables to continuous variables,

$\hat{q}_j \rightarrow \hat{\phi}(x)$, how does diagonalization change?

Recall equation (4.4),

$$\hat{q}'_j = \sum_k [\bar{O}]_{jk} \hat{q}_k$$

}

$$\hat{\phi}(x) = \int d^3y \, K(x,y) \hat{\phi}(y) \quad (4.13)$$

for some kernel function $K(x,y)$.

Recall: Feature of certain symmetric matrices:

If a matrix Q is Circulant,

that is, $[Q]_{jk} = [m]_{j-k}$

↑ some number $m \in \mathbb{R}$

$$\text{ie. } \bar{A} = \begin{pmatrix} 1 & 2 & 7 & \dots \\ 3 & 1 & 2 & 7 & \dots \\ & 3 & 1 & 2 & 7 & \dots \\ 0 & & 3 & 1 & 2 & \dots \\ & & & 3 & 1 & 2 & \dots \end{pmatrix} \quad (4.14)$$

then it is diagonalized by the Discrete Fourier Transform:

↑ complex entries

$$\bar{U} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \mu & \mu^2 & \dots & \mu^{n-1} \\ 1 & \mu^2 & \mu^4 & \dots & \mu^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & \dots & \dots & \mu^{jk} \end{pmatrix} \quad (4.15)$$

where $\mu = e^{\frac{2\pi i}{n}}$. $\mu^{jk} \leftarrow \mu^{jk} (jk)$ (4.16)

Thus, for continuous circulant matrices, we impose the (continuous) Fourier transform to diagonalize the matrix.

Discrete FT: $[\bar{O}]_{jk} = \frac{1}{n} e^{\frac{2\pi i}{n} \cdot jk}$ (4.17)

Continuous FT: $K(x,y) = \frac{1}{2\pi} e^{ixy}$ (4.18)

Hamiltonian Diagonalization: Gauss THESE ARE NOT HERMITIAN.

$$\hat{\phi}(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \hat{\phi}_{\vec{p}}(\vec{p}) \quad (4.19)$$

The argument of the field reveals whether we are in position or momentum space

Note that $\hat{\phi}_{\vec{p}}^{\dagger}(\vec{p}) = \hat{\phi}_{\vec{p}}(-\vec{p})$ (4.20).
↑ exercise (i)

Thus, eqⁿ (4.19) should diagonalize $\hat{H}_{KG}$.

* Construct Ladder Operators:

$$\hat{\phi}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} + \hat{a}_{\vec{p}}^{\dagger} e^{-i\vec{p} \cdot \vec{x}} \right) \quad (4.21)$$

$$\hat{\pi}(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} - \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}} \right) \quad (4.22)$$

and $\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$.

↑ exercise (i)
(ii)

Check Commutation Relations:

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \left(\frac{-i}{2} \right) \sqrt{\frac{\omega_{\vec{p}'}}{\omega_{\vec{p}}}} \left([\hat{a}_{-\vec{p}}^\dagger, \hat{a}_{\vec{p}'}] - [\hat{a}_{\vec{p}}, \hat{a}_{-\vec{p}'}^\dagger] \right) e^{i(\vec{p} \cdot \vec{x} + \vec{p}' \cdot \vec{y})} \quad (4.23)$$

↑ exercise...
(iii)

↑ arises from a change of variables.

↑ variable change
(?)^{*}

$$= i \delta^{(3)}(\vec{x} - \vec{y}).$$

* [Initially, this was ' $\vec{p}'$ ' not ' $-\vec{p}'$ ', but the initial given answer is inconsistent with the exercise, so I assumed there was a mistake and changed it to ' $-\vec{p}'$ '. See ex. (iii)]

This is solved by

$$[\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') \cdot \mathbb{I} \quad (4.24)$$

With this, the Klein-Gordon Hamiltonian (diagonalized) is

$$\hat{H}_{KG} = \int d^3 x \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} e^{i(\vec{p} + \vec{p}') \cdot \vec{x}} \left\{ -\frac{\sqrt{\omega_{\vec{p}} \omega_{\vec{p}'}}}{4} (\hat{a}_{\vec{p}} - \hat{a}_{-\vec{p}}^\dagger)(\hat{a}_{\vec{p}'} - \hat{a}_{-\vec{p}'}^\dagger) + \frac{(-\vec{p} \cdot \vec{p}' + m^2)}{4\sqrt{\omega_{\vec{p}} \omega_{\vec{p}'}}} (\hat{a}_{\vec{p}'} + \hat{a}_{-\vec{p}}^\dagger)(\hat{a}_{\vec{p}} + \hat{a}_{-\vec{p}'}^\dagger) \right\} \quad (4.25)$$

↑ exercise...
(iv)

$$= \int \frac{d^3 p}{(2\pi)^3} \omega_{\vec{p}} \left(\hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}} + \frac{1}{2} [\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}}^\dagger] \right) \quad (4.26)$$

$$\text{physically} \quad \text{Equivalent} \int \frac{d^3 p}{(2\pi)^3} \omega_{\vec{p}} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}} \quad \downarrow \delta^{(3)}(\dots) \quad (+ \infty \cdot \mathbb{I}) \quad (4.27)$$

but this never appears in any kind of real calculation/observation...

Exercises:

(i) $\hat{\phi}_p^\dagger(\vec{p}) = \hat{\phi}_p(-\vec{p})$. Proof: First note that

$$\hat{\phi}(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \hat{\phi}_p(\vec{p}).$$

The corresponding inverse Fourier Transform is then

$$\hat{\phi}_p(\vec{p}) = \int \frac{d^3 x}{(2\pi)^3} e^{-i\vec{p} \cdot \vec{x}} \hat{\phi}(\vec{x}).$$

$$\text{But } \hat{\phi}_p(-\vec{p}) = \int \frac{d^3 x}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \hat{\phi}(\vec{x}).$$

$$\text{and } \hat{\phi}_p^\dagger(\vec{p}) = \int \frac{d^3 x}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \hat{\phi}^\dagger(\vec{x})$$

But the $\hat{\phi}(\vec{x})$ operators are Hermitian ($\hat{\phi}(\vec{x}) = \hat{\phi}^\dagger(\vec{x})$)

hence $\hat{\phi}_p^\dagger(\vec{p}) = \hat{\phi}_p(-\vec{p})$ as required.



(ii) Ladder operator field construction:

$$\hat{\phi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(\hat{a}_p e^{i\vec{p}\cdot\vec{x}} + \hat{a}_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\hat{\pi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(\hat{a}_p e^{i\vec{p}\cdot\vec{x}} - \hat{a}_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right).$$

"Proof": We desire to determine continuous field operators from discrete analogues. First, note that

$$\left. \begin{aligned} \hat{q}_x' &= \frac{1}{\sqrt{2m\omega_x}} (\hat{a}_x + \hat{a}_x^\dagger) \\ \hat{p}_x' &= \frac{1}{i} \sqrt{\frac{m\omega_x}{2}} (\hat{a}_x - \hat{a}_x^\dagger) \end{aligned} \right\} (*)$$

in the discrete regime. In the continuous regime, $x \rightarrow \vec{x}$, and by the diagonalization principle (4.19), the raising/lowering operators must be diagonalized individually, hence

$$\begin{aligned} \hat{a}_{(x)} &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \hat{a}_p \quad \text{and} \\ \hat{a}_{(x)}^\dagger &= \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} \hat{a}_p^\dagger. \end{aligned}$$

Thus

$$\hat{q}_x' \rightarrow \hat{\phi}(\vec{x}) = \frac{1}{\sqrt{2\omega_x}} (\hat{a}_{(x)} + \hat{a}_{(x)}^\dagger)$$

$$\hat{p}_x' \rightarrow \hat{\pi}(\vec{x}) = (-i) \sqrt{\frac{\omega_x}{2}} (\hat{a}_{(x)} - \hat{a}_{(x)}^\dagger).$$

By diagonalizing into the momentum regime

(Fourier Transforming each operator individually),

$$\hat{\phi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(e^{i\vec{p}\cdot\vec{x}} \hat{a}_{\vec{p}} + e^{-i\vec{p}\cdot\vec{x}} \hat{a}_{\vec{p}}^\dagger \right)$$

$$\hat{\pi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(e^{i\vec{p}\cdot\vec{x}} \hat{a}_{\vec{p}} - e^{-i\vec{p}\cdot\vec{x}} \hat{a}_{\vec{p}}^\dagger \right),$$

as desired. ■

(iii) Commutation Relations

$$[\hat{\phi}(x), \hat{\pi}(y)] = \int \frac{d^3p d^3p'}{(2\pi)^6} \left(-\frac{i}{2} \right) \sqrt{\frac{\omega_p}{\omega_{p'}}} \left([\hat{a}_{-\vec{p}}, \hat{a}_{\vec{p}'}] - [\hat{a}_{\vec{p}}, \hat{a}_{-\vec{p}'}] \right) e^{i(\vec{p} \cdot \vec{x} + \vec{p}' \cdot \vec{y})}$$

Proof: The field diagonalizations are previously given from (ii)

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} + \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}} \right)$$

$$\hat{\pi}(y) = \int \frac{d^3p'}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{p'}}{2}} \left(\hat{a}_{\vec{p}'} e^{i\vec{p}' \cdot \vec{y}} - \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}' \cdot \vec{y}} \right).$$

Then

$$\begin{aligned} [\hat{\phi}(x), \hat{\pi}(y)] &= \hat{\phi}(x) \hat{\pi}(y) - \hat{\pi}(y) \hat{\phi}(x) \\ &= \int \frac{d^3p d^3p'}{(2\pi)^6} (-i) \sqrt{\frac{\omega_p}{4\omega_{p'}}} \left(\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} + \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}} \right) \\ &\quad \cdot \left(\hat{a}_{\vec{p}'} e^{i\vec{p}' \cdot \vec{y}} - \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}' \cdot \vec{y}} \right) \\ &\quad - \int \frac{d^3p d^3p'}{(2\pi)^6} (-i) \sqrt{\frac{\omega_p}{4\omega_{p'}}} \left(\hat{a}_{\vec{p}'} e^{i\vec{p}' \cdot \vec{y}} - \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}' \cdot \vec{y}} \right) \\ &\quad \cdot \left(\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} + \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}} \right) \\ &= \int \frac{d^3p d^3p'}{(2\pi)^6} \left(-\frac{i}{2} \right) \sqrt{\frac{\omega_p}{\omega_{p'}}} \left(\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} \hat{a}_{\vec{p}'} e^{i\vec{p}' \cdot \vec{y}} \right. \end{aligned}$$

$$\begin{aligned}
& - \hat{a}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}'\cdot\vec{z}} + \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \hat{a}_{\vec{p}'} e^{i\vec{p}'\cdot\vec{z}} \\
& - \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}'\cdot\vec{z}} - \hat{a}_{\vec{p}'} e^{i\vec{p}'\cdot\vec{z}} \hat{a}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} \\
& - \hat{a}_{\vec{p}'} e^{i\vec{p}'\cdot\vec{z}} \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} + \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}'\cdot\vec{z}} \hat{a}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} \\
& + \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}'\cdot\vec{z}} \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \Big) \\
& = \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \left(\frac{-i}{2} \right) \sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{p}'}}} \Big([\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}'}] e^{i(\vec{p}\cdot\vec{x} + \vec{p}'\cdot\vec{z})} \\
& + [\hat{a}_{\vec{p}'}^\dagger, \hat{a}_{\vec{p}}] e^{i(\vec{p}\cdot\vec{x} - \vec{p}'\cdot\vec{z})} + [\hat{a}_{\vec{p}}^\dagger, \hat{a}_{\vec{p}'}] e^{i(-\vec{p}\cdot\vec{x} + \vec{p}'\cdot\vec{z})} \\
& + [\hat{a}_{\vec{p}'}^\dagger, \hat{a}_{\vec{p}}^\dagger] e^{-i(\vec{p}\cdot\vec{x} + \vec{p}'\cdot\vec{z})} \Big)
\end{aligned}$$

But since $[\hat{a}_{\vec{f}}, \hat{a}_{\vec{g}}] = [\hat{a}_{\vec{f}}^\dagger, \hat{a}_{\vec{g}}^\dagger] = 0$, we have

$$\begin{aligned}
& = \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \left(\frac{-i}{2} \right) \sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{p}'}}} \Big([\hat{a}_{\vec{p}'}^\dagger, \hat{a}_{\vec{p}}] e^{i(\vec{p}\cdot\vec{x} - \vec{p}'\cdot\vec{z})} \\
& + [\hat{a}_{\vec{p}}^\dagger, \hat{a}_{\vec{p}'}] e^{i(-\vec{p}\cdot\vec{x} + \vec{p}'\cdot\vec{z})} \Big).
\end{aligned}$$

Since the first term goes into

$$[\hat{a}_{\vec{p}'}^\dagger, \hat{a}_{\vec{p}}] = -[\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}'}^\dagger],$$

and by invoking a variable change

$$\vec{p} \rightarrow -\vec{p}$$

in the latter term and

$$\vec{p}' \rightarrow -\vec{p}'$$

in the first term, we have that

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \left(\frac{-i}{2} \right) \sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{p}'}}} \left([\hat{a}_{-\vec{p}}^{\dagger}, \hat{a}_{\vec{p}'}] e^{i(\vec{p} \cdot \vec{x} + \vec{p}' \cdot \vec{z})} - [\hat{a}_{\vec{p}}, \hat{a}_{-\vec{p}'}^{\dagger}] e^{i(\vec{p} \cdot \vec{x} + \vec{p}' \cdot \vec{z})} \right).$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \left(\frac{-i}{2} \right) \sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{p}'}}} \left([\hat{a}_{-\vec{p}}^{\dagger}, \hat{a}_{\vec{p}'}] - [\hat{a}_{\vec{p}}, \hat{a}_{-\vec{p}'}^{\dagger}] \right) \cdot e^{i(\vec{p} \cdot \vec{x} + \vec{p}' \cdot \vec{z})}$$

as desired. ■

(iv) Klein-Gordon Diagonalization

$$\hat{H}_{KG} = \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(\vec{p} + \vec{p}') \cdot \vec{x}} \left\{ -\frac{\sqrt{\omega_{\vec{p}} \omega_{\vec{p}'}}}{4} (\hat{a}_{\vec{p}} - \hat{a}_{-\vec{p}}^\dagger)(\hat{a}_{\vec{p}'} - \hat{a}_{-\vec{p}'}^\dagger) \right. \\ \left. + \frac{(-\vec{p} \cdot \vec{p}' + m^2)}{4\sqrt{\omega_{\vec{p}} \omega_{\vec{p}'}}} (\hat{a}_{\vec{p}} + \hat{a}_{-\vec{p}}^\dagger)(\hat{a}_{\vec{p}'} + \hat{a}_{-\vec{p}'}^\dagger) \right\}.$$

Proof: The Klein-Gordon Hamiltonian, for fields, is given by

$$\hat{H}_{KG} = \frac{1}{2} \int d^3x \left\{ \hat{\pi}^2(x) + (\nabla \hat{\phi}(x))^2 + m^2 \hat{\phi}^2(x) \right\}$$

With the field operators, previously given as

$$\hat{\phi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} + \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}} \right)$$

$$\hat{\pi}(\vec{y}) = \int \frac{d^3p'}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}'}}{2}} \left(\hat{a}_{\vec{p}'} e^{i\vec{p}' \cdot \vec{y}} - \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}' \cdot \vec{y}} \right). \\ (\vec{z} \rightarrow \vec{x})$$

First, note that due to the Hamiltonian density, we must integrate over all space $\int d^3x$ (that is why that is there).

Secondly, we must understand that

$$\begin{aligned}\hat{\phi}^2(\vec{x}) &= \hat{\phi}_{\vec{p}}(\vec{x}) \cdot \hat{\phi}_{\vec{p}'}(\vec{x}) \\ &= \left\{ \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} (\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} + \hat{a}_{\vec{p}}^{\dagger} e^{-i\vec{p} \cdot \vec{x}}) \right\} \\ &\quad \cdot \left\{ \int \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}'}}} (\hat{a}_{\vec{p}'} e^{i\vec{p}' \cdot \vec{x}} + \hat{a}_{\vec{p}'}^{\dagger} e^{-i\vec{p}' \cdot \vec{x}}) \right\}\end{aligned}$$

(each field observable corresponds to its own momentum space, since this is just the square magnitude of the field operator)

And this is equivalent to $\hat{\pi}^2(\vec{x})$.

The gradient $\nabla \hat{\phi}$ only acts on the $e^{i\vec{p} \cdot \vec{x}}$ terms, thus

$$\nabla \hat{\phi} = \int \frac{d^3 p}{(2\pi)^3} \frac{i\vec{p}}{\sqrt{2\omega_{\vec{p}}}} (\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} - \hat{a}_{\vec{p}}^{\dagger} e^{-i\vec{p} \cdot \vec{x}})$$

Therefore, we have in combination that

$$\hat{H}_{KG} = \frac{1}{2} \int d^3 x \int \frac{d^3 p d^3 p'}{(2\pi)^6} \left\{ -\frac{\sqrt{\omega_{\vec{p}} \omega_{\vec{p}'}}}{2} (\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} - \hat{a}_{\vec{p}}^{\dagger} e^{-i\vec{p} \cdot \vec{x}}) \right.$$

$$\begin{aligned}
& \cdot (\hat{a}_{\vec{p}'} e^{i\vec{p}' \cdot \vec{x}} - \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}' \cdot \vec{x}}) \\
& + \frac{-1}{2\sqrt{\omega_{\vec{p}} \omega_{\vec{p}'}}} (\bar{p} \hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} - \bar{p} \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}}) \\
& \quad \cdot (\bar{p}' \hat{a}_{\vec{p}'} e^{i\vec{p}' \cdot \vec{x}} - \bar{p}' \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}' \cdot \vec{x}}) \\
& + \frac{m^2}{2\sqrt{\omega_{\vec{p}} \omega_{\vec{p}'}}} (\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} + \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}}) \\
& \quad \cdot (\hat{a}_{\vec{p}'} e^{i\vec{p}' \cdot \vec{x}} + \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}' \cdot \vec{x}}) \}
\end{aligned}$$

And by invoking a variable change on the Hermitian conjugate terms

$$\begin{aligned}
\hat{H}_{KG} = \int d^3x \frac{d^3p d^3p'}{(2\pi)^6} & \left\{ -\frac{\sqrt{\omega_{\vec{p}} \omega_{\vec{p}'}}}{4} (\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} - \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}}) \cdot (\hat{a}_{\vec{p}'} e^{i\vec{p}' \cdot \vec{x}} - \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}' \cdot \vec{x}}) \right. \\
& - \frac{1}{4\sqrt{\omega_{\vec{p}} \omega_{\vec{p}'}}} (\bar{p} \hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} + \bar{p} \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}}) \\
& \quad \cdot (\bar{p}' \hat{a}_{\vec{p}'} e^{i\vec{p}' \cdot \vec{x}} + \bar{p}' \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}' \cdot \vec{x}}) \\
& + \frac{m^2}{4\sqrt{\omega_{\vec{p}} \omega_{\vec{p}'}}} (\hat{a}_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} + \hat{a}_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}}) \\
& \quad \cdot (\hat{a}_{\vec{p}'} e^{i\vec{p}' \cdot \vec{x}} + \hat{a}_{\vec{p}'}^\dagger e^{-i\vec{p}' \cdot \vec{x}}) \}
\end{aligned}$$

and by factoring out a $e^{i(\vec{p} + \vec{p}') \cdot \vec{x}}$ and collecting like terms, we find that

$$\hat{H}_{KG} = \int d^3x \frac{d^3p d^3p'}{(2\pi)^6} e^{i(\vec{p} + \vec{p}') \cdot \vec{x}} \left\{ -\frac{\sqrt{\omega_{\vec{p}} \omega_{\vec{p}'}}}{4} (\hat{a}_{\vec{p}} - \hat{a}_{-\vec{p}}^\dagger) (\hat{a}_{\vec{p}'} - \hat{a}_{-\vec{p}'}^\dagger) + \frac{-\vec{p} \cdot \vec{p}' + m^2}{4 \sqrt{\omega_{\vec{p}} \omega_{\vec{p}'}}} (\hat{a}_{\vec{p}} + \hat{a}_{-\vec{p}}^\dagger) (\hat{a}_{\vec{p}'} + \hat{a}_{-\vec{p}'}^\dagger) \right\}$$

as desired. ■

